Hankel hyperdeterminants, rectangular Jack polynomials and even powers of the Vandermonde

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Abstract

We investigate the link between rectangular Jack polynomials and Hankel hyperdeterminants. As an application we give an expression of the even power of the Vandermonde in term of Jack polynomials.

1   Introduction

Few after he introduced the modern notation for determinants [2], Cayley proposed several extensions to higher dimensional arrays under the same name *hyperdeterminant* [3, 4]. The notion considered here is apparently the simplest one, defined for a kth order tensor $\mathbf{M} = (M_{i_1\cdots i_k})_{1 \leq i_1,...,i_k \leq n}$ on an n-dimensional space by

$$\mathrm{Det}\, M = \frac{1}{n!} \sum_{\sigma=(\sigma_1,\cdots,\sigma_k)\in\mathfrak{S}_n^k} \mathrm{sign}(\sigma) M^\sigma,$$

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where $\text{sign}(\sigma) = \text{sign}(\sigma_1) \cdots \text{sign}(\sigma_k)$ is the product of the signs of the permutations, $M^\sigma = M_{\sigma_1(1)\dots\sigma_k(1)} \cdots M_{\sigma_1(n)\dots\sigma_k(n)}$ and $\mathfrak{S}_n$ denotes the symmetric group. Note that others hyperdeterminants are found in literature. For example, those considered by Gelfand, Kapranov and Zelevinsky [7] are bigger polynomials with geometric properties. Our hyperdeterminant is a special case of the *riciens* [26, 21] with only alternant indices. For an hypermatrice with an even number of indices, it generates the space of the polynomial invariants of lowest degree. Easily, one obtains the nullity of Det when k is odd and its invariance under the action of the group $SL_n^{\times k}$. Few references exists on the topics see [26, 29, 30, 11, 1, 8] and before [22] the multidimensional analogues of Hankel determinant do not seem to have been investigated.

In this paper, we discuss about the links between Hankel hyperdeterminsants and Jack's symmetric functions indexed by rectangular partitions. Jack's symmetric functions are a one parameter (denoted by α in this paper) generalization of Schur functions. They were defined by Henry Jack in 1969 in the aim to interpolate between Schur fonctions ($\alpha = 1$) and zonal polynomials ($\alpha = 2$) [15, 16]. The story of Jack's polynomials is closely related to the generalizations of the Selberg integral [1, 12, 14, 19, 27, 22]. The relation between Jack's polynomials and hyperdeterminants appeared implicitly in this context in [22], when one of the author with J.-Y. Thibon gave an expression of the Kaneko integral [12] in terms of Hankel hyperdeterminant. More recently, Matsumoto computed [25] an hyperdeterminantal Jacobi-Trudi type formula for rectangular Jack polynomials.

The paper is organized as follow. In Section 2, after we recall definitions of Hankel and Toeplitz hyperdeterminants, we explain that an Hankel hyperdeterminant can be viewed as the *umber* of an even power of the Vandermonde via the substitution $\int_\mathbb{Y} : x^n \to \Lambda^n(\mathbb{Y})$ where $\Lambda^n(\mathbb{Y})$ denotes the nth elementary symmetric functions on the alphabet $\mathbb{Y}$. Section 3 is devoted to the generalization of the Matsumoto formula [25] to *almost rectangular* Jack polynomials. In Section 4, we give an equality involving the substitution $\int_\mathbb{Y}$ and skew Jack polynomials. As an application, we give in Section 5 expressions of even powers of the Vandermonde determinant in terms of Schur functions and Jack polynomials.

2   Hankel and Toeplitz Hyperdeterminants of symmetric functions

2.1   Symmetric functions

Symmetric functions over an alphabet $\mathbb{X}$ are functions which are invariant under permutation of the variables. The $\mathbb{C}$-space of the symmetric functions over $\mathbb{X}$ is an algebra which will be denoted by $Sym(\mathbb{X})$.

Let us consider the complete symmetric functions whose generating series is

$$\sigma_t(\mathbb{X}) := \sum_i S^i(\mathbb{X})t^i = \prod_{x \in \mathbb{X}} \frac{1}{1 - xt},$$

the elementary symmetric functions

$$\lambda_t(\mathbb{X}) := \sum_i \Lambda^i(\mathbb{X})t^i = \prod_{x \in \mathbb{X}}(1 + xt) = \sigma_{-1}(\mathbb{X})^{-1},$$

and power sum symmetric functions

$$\psi_t(\mathbb{X}) := \sum_i \Psi^i(\mathbb{X})\frac{t^i}{i} = \log(\sigma_t(\mathbb{X})).$$

When there is no algebraic relation between the letters of $\mathbb{X}$, $Sym(\mathbb{X})$ is a free (associative, commutative) algebra over complete, elementary or power sum symmetric functions

$$Sym = \mathbb{C}[S^1, S^2, \cdots] = \mathbb{C}[\Lambda^1, \Lambda^2, \cdots] = \mathbb{C}[\Psi^1, \Psi^2, \cdots].$$

As a consequence, the algebra $Sym(\mathbb{X})$ ($\mathbb{X}$ being infinite or not) is spanned by the set of the decreasing products of the generators

$$S^\lambda = S^{\lambda_n} \dots S^{\lambda_1}, \ \Lambda^\lambda = \Lambda^{\lambda_n} \dots \Lambda^{\lambda_1}, \ \Psi^\lambda = \Psi^{\lambda_n} \dots \Psi^{\lambda_1},$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ is a (decreasing) partition.

The algebra $Sym(\mathbb{X})$ admits also non multiplicative basis. For example, the monomial functions defined by

$$m_\lambda(\mathbb{X}) = \sum x_{i_1}^{\lambda_1} \cdots x_{i_n}^{\lambda_n},$$

where the sum is over all the distinct monomials $x_{i_1}^{\lambda_1} \cdots x_{i_n}^{\lambda_n}$ with $x_{i_1}, \ldots, x_{i_n} \in \mathbb{X}$, and the Schur functions defined via the Jacobi-Trudi formula

$$S_\lambda(\mathbb{X}) = \det(S^{\lambda_i - i + j}(\mathbb{X})). \tag{1}$$

Note that the Schur basis admits other alternative definitions. For example, it is the only basis such that

1. It is orthogonal for the scalar product defined on power sums by

$$\langle \Psi_\lambda, \Psi_\mu \rangle = \delta_{\lambda,\mu} z_\lambda \tag{2}$$

   where $\delta_{\mu,\nu}$ is the Kronecker symbol (equal to 1 if $\mu = \nu$ and 0 otherwise) and $z_\lambda = \prod_i i^{m_i(\lambda)} m_i(\lambda)!$ if $m_i(\lambda)$ denotes the multiplicity of i as a part of λ.

2. The coefficient of the dominant term in the expansion in the monomial basis is 1,

$$S_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu.$$

When $\mathbb{X} = \{x_1, \ldots, x_n\}$ is finite, a Schur function has another determinantal expression

$$S_\lambda(\mathbb{X}) = \frac{\det(x_i^{\lambda_j + n - j})}{\Delta(\mathbb{X})},$$

where $\Delta(\mathbb{X}) = \prod_{i<j}(x_i - x_j)$ denotes the Vandermonde determinant.

2.2 Definitions and General properties

A Hankel hyperdeterminant is an hyperdeterminant of a tensor whose entries depend only of the sum of the indices $M_{i_1, \ldots, i_{2k}} = f(i_1 + \cdots + i_{2k})$. One of the authors investigated such polynomials in relation with the Selberg integral [22, 23].

Without lost of generality, we will consider the polynomials

$$\mathfrak{H}_n^k(\mathbb{X}) = \mathrm{Det}\left(\Lambda^{i_1 + \cdots + i_{2k}}(\mathbb{X})\right)_{0 \leq i_1, \ldots, i_{2k} \leq n - 1}, \tag{3}$$

where $\Lambda_m(\mathbb{X})$ is the mth elementary function on the alphabet $\mathbb{X}$.

Let us consider a shifted version of Hankel hyperdeterminants

$$\mathfrak{H}_v^k(\mathbb{X}) = \mathrm{Det}\left(\Lambda^{i_1 + \cdots + i_{2k} + v_{i_1}}(\mathbb{X})\right)_{0 \leq i_1, \ldots, i_{2k} \leq n - 1}. \tag{4}$$

where $v = (v_0, \ldots, v_{n-1}) \in \mathbb{Z}^n$. Note that (3) implies $M_{0,\ldots,0} = \Lambda_0(\mathbb{X}) = 1$ by convention. But if $M_{0,\ldots,0} \neq 0, 1$, this property can be recovered using a suitable normalization and, if $M_{0\ldots0} = 0$, by using the shifted version (4) of the Hankel hyperdeterminant. As in [25], one defines Toëplitz hyperdeterminant by giving directly the shifting version

$$\mathfrak{T}_v^k(\mathbb{X}) = \text{Det}\left(\Lambda^{i_1+\cdots+i_k-(i_{k+1}+\cdots+i_{2k})+v_{i_1}}(\mathbb{X})\right)_{0 \leq i_1,\ldots,i_{2k} \leq n-1}. \tag{5}$$

Toëplitz hyperdeterminants are related to Hankel hyperdeterminants by the following formulae.

Proposition 2.1   *1. $\mathfrak{H}_v^k(\mathbb{X}) = (-1)^{\frac{kn(n-1)}{2}} \mathfrak{T}_{v+(k(n-1))^n}^k(\mathbb{X})$*

2. $\mathfrak{T}_v^k(\mathbb{X}) = (-1)^{\frac{kn(n-1)}{2}} \mathfrak{H}_{v+(k(1-n))^n}^k(\mathbb{X})$.

Proof The equalities (1) and (2) are equivalent and are direct consequences of the definitions (4) and (5),

$$
\begin{aligned}
\mathfrak{T}_v^k(\mathbb{X}) &= \text{Det}\left(\Lambda^{i_1+\cdots+i_k+(n-1-i_{k+1}+\cdots+(n-1-i_{2k}-k(n-1))}(\mathbb{X})\right) \\
&= (-1)^{\frac{kn(n-1)}{2}} \mathfrak{H}_{v+(k(1-n))^n}(\mathbb{X}).
\end{aligned}
$$

$\square$.

2.3   The substitution $x^n \to \Lambda^n(\mathbb{Y})$

Let $\mathbb{X} = \{x_1, \ldots, x_n\}$ be a finite alphabet and $\mathbb{Y}$ be another (potentially infinite) alphabet. For simplicity we will denote by $\int_{\mathbb{Y}}$ the substitution

$$\int_{\mathbb{Y}} x^p = \Lambda^p(\mathbb{Y}),$$

for each $x \in \mathbb{X}$ and each $p \in \mathbb{Z}$.

    The main tool of this paper is the following proposition.

Proposition 2.2 *For any integer $k \in \mathbb{N} - \{0\}$, one has*

$$\frac{1}{n!} \int_{\mathbb{Y}} \Delta(\mathbb{X})^{2k} = \mathfrak{H}_n^k(\mathbb{Y})$$

where $\Delta(\mathbb{X}) = \prod_{i<j}(x_i - x_j)$.

Proof It suffices to develop the power of the Vandermonde determinant

$$\Delta(\mathbb{X})^{2k} \;=\; \det\left(x_i^{j-1}\right)^{2k} \;=\; \sum_{\sigma_1,\cdots,\sigma_{2k}\in\mathfrak{S}_n} \operatorname{sign}(\sigma_1\cdots\sigma_{2k})\prod_i x_i^{\sigma_1(i)+\cdots+\sigma_{2k}(i)-2k}.$$

Hence, applying the substitution, one obtains

$$
\begin{aligned}
\frac{1}{n!}\int_{\mathbb{Y}}\Delta(\mathbb{X})^{2k} &= \frac{1}{n!}\sum_{\sigma_1,\cdots,\sigma_{2k}\in\mathfrak{S}_n}\operatorname{sign}(\sigma_1\cdots\sigma_{2k})\prod_i \Lambda^{\sigma_1(i)+\cdots+\sigma_{2k}(i)-2k}(\mathbb{Y})\\
&= \mathfrak{H}_n^k(\mathbb{Y}).
\end{aligned}
$$

$\square$

More generally, the Jacobi-Trudi formula (1) implies the following result.

Proposition 2.3 *One has*

$$\frac{1}{n!}\int_{\mathbb{Y}} S_\lambda(\mathbb{X})\Delta(\mathbb{X})^{2k} = \mathfrak{H}^k_{\operatorname{reverse}_n(\lambda)}(\mathbb{Y}),$$

where $\operatorname{reverse}_n(v) = (v_n,\ldots,v_1)$ *if* $v = (v_1,\ldots,v_p)$ *is a composition with* $p \le n$ *and* $v_{p+i} = 0$ *for* $1 \le i \le n-p$.

Proof It suffices to remark that

$$S_\lambda(\mathbb{X})\Delta(\mathbb{X})^{2k} = \det(x_i^{\lambda_{n-j+1}+j-1})\det\left(x_i^{j-1}\right)^{2k-1},$$

and apply the same computation than in the proof of Proposition 2.2. $\square$

Example 2.4 *If* $k = 1$ *then using the second Jacobi-Trudi formula*

$$S_\lambda = \det(\Lambda^{\lambda'_{n-i}-i+j}) \tag{6}$$

where λ' *denotes the conjugate partition of* λ, *Proposition 2.3 implies*

$$\frac{1}{n!}\int_{\mathbb{Y}} S_\lambda(\mathbb{X})\Delta(\mathbb{X})^2 = (-1)^{\frac{n(n-1)}{2}} S_{(\lambda+(n-1)^n)'}(\mathbb{Y}).$$

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3 Jack Polynomials and Hyperdeterminants

In this section, we will consider the symmetric functions as a λ-ring endowed with the operator S^i, and we will use the definition of addition and multiplication of alphabets in this context (see *e.g.* [18]). Let $\mathbb{X}$ and $\mathbb{Y}$ be two alphabets, the symmetric functions over the alphabet $\mathbb{X} + \mathbb{Y}$ are generated by the complete functions $S^i(\mathbb{X}+\mathbb{Y})$ defined by $\sigma_t(\mathbb{X}+\mathbb{Y}) = \sigma_t(\mathbb{X})\sigma_t(\mathbb{Y}) = \sum_i S^i(\mathbb{X}+\mathbb{Y})t^i$. If $\mathbb{X} = \mathbb{Y}$, one has $\sigma_t(2\mathbb{X}) := \sigma_t(\mathbb{X} + \mathbb{X}) = \sigma_t(\mathbb{X})^2$. Similarly one defines $\sigma_t(\alpha\mathbb{X}) = \sigma_t(\mathbb{X})^\alpha$. In particular, the equality $\sigma_t(-\mathbb{X}) = \prod_x (1 - xt) = \lambda_{-t}(\mathbb{X})$ gives $S^i(-\mathbb{X}) = (-1)^i \Lambda^i(\mathbb{X})$. The product of two alphabet $\mathbb{X}$ and $\mathbb{Y}$ is defined by $\sigma_t(\mathbb{X}\mathbb{Y}) = \sum S^i(\mathbb{X}\mathbb{Y})t^i = \prod_{x \in \mathbb{X}} \prod_{y \in \mathbb{Y}} \frac{1}{1-xyt}$. Note that $\sigma_1(\mathbb{X}\mathbb{Y}) = K(\mathbb{X}, \mathbb{Y}) = \sum_\lambda S_\lambda(\mathbb{X})S_\lambda(\mathbb{Y})$ is the Cauchy Kernel.

3.1 Jack polynomials

One considers a one parameter generalization of the scalar product (2) defined by $\langle \Psi_\lambda, \Psi_\mu \rangle_\alpha = \delta_{\lambda,\mu} z_\lambda \alpha^{l(\lambda)}$, where $l(\lambda) = n$ denotes the length of the partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ with $\lambda_n > 0$. The Jack polynomials $P_\lambda^{(\alpha)}$ are the unique symmetric functions orthogonal for $\langle \, , \, \rangle_\alpha$ and such that $P_\lambda^{(\alpha)} = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu}^{(\alpha)} m_\mu$. Note that in the case when $\alpha = 1$, one recovers the definition of Schur functions, *i.e.* $P_\lambda^{(1)} = S_\lambda$. Let $(Q_\lambda^{(\alpha)})$ be the dual basis of $(P_\lambda^{(\alpha)})$. The polynomials $P_\lambda^{(\alpha)}$ and $Q_\lambda^{(\alpha)}$ are equal up to a scalar factor and the coefficient of proportionality is computed explicitly in [24] VI. 10:

$$b_\lambda^{(\alpha)} := \frac{P_\lambda^{(\alpha)}}{Q_\lambda^{(\alpha)}} = \langle P_\lambda^{(\alpha)}, P_\lambda^{(\alpha)} \rangle^{-1} = \prod_{(i,j) \in \lambda} \frac{\alpha(\lambda_i - j) + \lambda_j' - i + 1}{\alpha(\lambda_i - j + 1) + \lambda_j' - i}. \qquad (7)$$

Let $\mathbb{X} = \{x_1, \cdots, x_n\}$ be a finite alphabet and denote by $\mathbb{X}^\vee$ the alphabet of the inverse $\{x_1^{-1}, \ldots, x_n^{-1}\}$. Let us introduce the second scalar product by

$$\langle f, g \rangle_{n,\alpha}' = \frac{1}{n!} \text{C.T.} \{f(\mathbb{X})g(\mathbb{X}^\vee) \prod_{i \neq j} (1 - x_i x_j^{-1})^{\frac{1}{\alpha}}\},$$

see [24] VI. 10. The polynomials $P_\lambda^{(\alpha)}$ and $Q_\lambda^{(\alpha)}$ are also orthogonal for this scalar product.

For simplicity, we will consider also another normalisation defined by

$$R_\lambda^{(\alpha),n}(\mathbb{Y}) := \langle P_{\lambda'}^{(1/\alpha)}, Q_{\lambda'}^{(1/\alpha)} \rangle_{n,\frac{1}{\alpha}}' Q_\lambda^{(\alpha)}(\mathbb{Y}).$$

Note that the polynomial $R_\lambda^{(\alpha),n}$ is not zero only when $l(\lambda) \leq n$ and in this case the value of the coefficient $\langle P_{\lambda'}^{(1/\alpha)}, Q_{\lambda'}^{(1/\alpha)} \rangle'_{n,\frac{1}{\alpha}}$ is known to be

$$\langle P_{\lambda'}^{(1/\alpha)}, Q_{\lambda'}^{(1/\alpha)} \rangle'_{n,\frac{1}{\alpha}} = \frac{1}{n!} \prod_{(i,j) \in \lambda'} \frac{n + \frac{1}{\alpha}(j-1) - i + 1}{n + \frac{j}{\alpha} - i} \text{C.T.} \left\{ \prod_{i \neq j} \left(1 - x_i x_j^{-1} \right)^\alpha \right\}$$
(8)

see [24] VI 10.

3.2 The operator $\int_{\mathbb{Y}}$ and almost rectangle Jack polynomials

Suppose that $\mathbb{X} = \{x_1, \ldots, x_n\}$ is a finite alphabet. Let $\mathbb{Y} = \{y_1, \cdots\}$ be another (potentially infinite) alphabet and consider the integral

$$I_{n,k}(\mathbb{Y}) = \frac{1}{n!} \text{C.T.} \{\Lambda^n(\mathbb{X}^\vee)^{p+k(n-1)} \Lambda^l(\mathbb{X}^\vee) \prod (1 + x_i y_j) \Delta(\mathbb{X})^{2k}\}. \quad (9)$$

Note that,

$$I_{n,k}(\mathbb{Y}) = \frac{(-1)^{\frac{kn(n-1)}{2}}}{n!} \text{C.T.} \{\Lambda^n(\mathbb{X}^\vee)^p \Lambda^l(\mathbb{X}^\vee) \prod (1 + x_i y_j) \prod_{i \neq j} (1 - \frac{x_i}{x_j})^k\}. \quad (10)$$

But

$$\Lambda^n(\mathbb{X}^\vee)^p \Lambda^l(\mathbb{X}^\vee) = P_{(p+1)^l p^{n-l}}^{(1/k)}(\mathbb{X}^\vee), \quad (11)$$

and

$$\prod (1 + x_i y_j) = \sum_\lambda Q_\lambda^{(1/k)}(\mathbb{X}) Q_{\lambda'}^{(k)}(\mathbb{Y}). \quad (12)$$

Hence, from the orthogonality of $P_\lambda^{(\alpha)}$ and $Q_\lambda^{(\alpha)}$, equalities (10), (11) and (12) imply

$$I_{n,k}(\mathbb{Y}) = (-1)^{\frac{kn(n-1)}{2}} R_{n^p l}^{(k),n}(\mathbb{Y}). \quad (13)$$

On the other hand, one has the equality

$$\int_{\mathbb{Y}} x^m = \Lambda_m(\mathbb{Y}) = \text{C.T.} \{x^{-m} \prod_i (1 + x y_i)\}.$$

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Remarking that $\Delta(\mathbb{X}^{\vee}) = (-1)^{\frac{n(n-1)}{2}} \frac{\Delta(\mathbb{X})}{\Lambda^n(\mathbb{X})^{n-1}}$, and $\Lambda^n(\mathbb{X})^m \Lambda^l(\mathbb{X}) = S_{(m^n)+(1^l)}(\mathbb{X})$ for each $m \in \mathbb{Z}$, Equality (9) can be written as

$$I_{n,k}(\mathbb{Y}) = \frac{1}{n!} \int_{\mathbb{Y}} S_{((p-k(n-1))^n)+(1^l)}(\mathbb{X})\Delta(\mathbb{X})^{2k} = (-1)^{\frac{kn(n-1)}{2}} \mathfrak{T}^k_{p^{n-l}(p+1)^l}(\mathbb{X}).$$
(14)

One deduces an hyperdeterminantal expression for a Jack polynomial indexed by the partition $n^p l$.

Proposition 3.1 *For any positive integers n, p, l and k, one has.*

$$R^{(k),n}_{n^p l} = \mathfrak{T}^k_{p^{n-l}(p+1)^l}.$$

The constant term appearing in (8) is a special case of the the Dyson Conjecture [6]. The conjecture of Dyson has been proved the same year independently by Gunson [10] and Wilson [31] (in 1970 I. J. Good [9] have shown an elegant elementary proof involving Lagrange interpolation),

$$\text{C.T.} \prod_{i \neq j} \left(1 - x_i x_j^{-1}\right)^{a_i} = \binom{a_1 + \cdots + a_n}{a_1, \ldots, a_n},$$

for $a_1, \ldots, a_n \in \mathbb{N}$. Hence, one has

$$Q^{(k)}_{n^p l}(\mathbb{Y}) = n! \binom{kn}{k, \cdots, k}^{-1} \kappa(n, p, l; k)\mathfrak{T}^k_{p^{n-l}(p+1)^l}(\mathbb{Y})$$

where

$$\kappa(n, p, l; k) = \prod_{i=1}^{n} \prod_{j=1}^{p} \frac{j + k(i-1)}{j - 1 + ki} \prod_{i=1}^{l} \frac{p + 1 + k(n-i)}{p + k(n-i+1)}.$$

In particular, when $l = 0$, one recovers a theorem by Matsumoto.

Corollary 3.2 *(Matsumoto [25])*

$$P^{(k)}_{n^p}(\mathbb{Y}) = n! \binom{kn}{k, \ldots, k}^{-1} \mathfrak{T}^k_{p^n}(\mathbb{Y}).$$

Proof From equalities (8) and (7), one has

$$\langle P^{(1/k)}_{p^n}, Q^{(1/k)}_{p^n} \rangle'_{1/k,n} = \frac{1}{n!} \binom{kn}{k, \ldots, k} \langle P^{(k)}_{n^p l}, P^{(k)}_{n^p l} \rangle_k.$$

Applying Proposition 3.1, one finds the result. □

Setting $p = k(n-1)$, one obtains the expression of an Hankel hyperdeterminant as a Jack polynomials.

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Corollary 3.3

$$\mathfrak{H}_n^k(\mathbb{Y}) = \frac{(-1)^{\frac{kn(n-1)}{2}}}{n!} \binom{kn}{k,\ldots,k} P_{n^{k(n-1)}}^{(k)}(\mathbb{Y}).$$

3.3 Jack polynomials with parameter $\alpha = \frac{1}{k}$

Let $\mathbb{Y} = \{y_1, y_2, \cdots\}$ be a (potentially infinite alphabet). Consider the endomorphism defined on the power sums symmetric functions $\Psi_p(\mathbb{Y})$ by $\omega_\alpha(\Psi_p(\mathbb{Y})) := \Psi_p(-\alpha\overline{\mathbb{Y}}) = (-1)^{p-1}\alpha\Psi_p(\mathbb{Y})$ (see [24] VI 10), where $\overline{\mathbb{Y}} = \{-y_1, -y_2, \cdots\}$. This map is known to satisfy the identities

$$\omega_\alpha P_\lambda^{(\alpha)}(\mathbb{Y}) = Q_{\lambda'}^{(\frac{1}{\alpha})}(\mathbb{Y})$$

and

$$\omega_\alpha \Lambda^n(\mathbb{Y}) = g_{(\frac{1}{\alpha})}^n(\mathbb{Y}) := \Lambda^n(-\alpha\overline{\mathbb{Y}}).$$

Applying ω_k on Proposition 3.1, one obtains the expression of a Jack polynomial with parameter $\alpha = \frac{1}{k}$ for an almost rectangular shape $\lambda = (p+1)^l p^{n-l}$ as a shifted Toeplitz hyperdeterminant whose entries are

$$M_{i_1\ldots i_{2k}} = g_{\frac{1}{k}}^{i_1+\cdots+i_k-i_{k+1}-\cdots-i_{2k}+\lambda_{n-i_1+1}}.$$

Proposition 3.4 *One has*

$$P_{(p+1)^l p^{n-l}}^{(\frac{1}{k})}(\mathbb{Y}) = n! \binom{kn}{k,\ldots,k}^{-1} \kappa(n,p,l;k)\mathfrak{T}_{p^{n-l}(p+1)^l}^{(k)}(-k\overline{\mathbb{Y}}).$$

Proof It suffices to apply Proposition 3.1 with the alphabet $-\alpha\mathbb{Y}$ to find

$$Q_{n^p l}^{(k)}(-k\mathbb{Y}) = n! \binom{kn}{k,\ldots,k}^{-1} \kappa(n,p,l;k)\mathfrak{T}_{p^{n-l}(p+1)^l}^{(k)}(-k\overline{\mathbb{Y}}).$$

The result follows from

$$Q_{n^p l}^{(k)}(-k\overline{\mathbb{Y}}) = \omega_k Q_{n^p l}^{(k)}(\mathbb{Y}) = P_{(p+1)^l p^{n-l}}^{(\frac{1}{k})}(\mathbb{Y}).$$

$\square$

4 Skew Jack polynomials and Hankel hyper-determinants

4.1 Skew Jack polynomials

Let us define as in [24] VI 10, the skew Q functions by

$$\langle Q_{\lambda/\mu}^{(\alpha)}, P_{\nu}^{(\alpha)} \rangle := \langle Q_{\lambda}^{(\alpha)}, P_{\mu}^{(\alpha)} P_{\nu}^{(\alpha)} \rangle.$$

Straightforwardly, one has

$$Q_{\lambda/\mu}^{(\alpha)} = \sum_{\nu} \langle Q_{\lambda}^{(\alpha)}, P_{\nu}^{(\alpha)} P_{\mu}^{(\alpha)} \rangle Q_{\nu}^{(\alpha)}. \tag{15}$$

Classically, the skew Jack polynomials appear when one expands a Jack polynomial on a sum of alphabet.

Proposition 4.1 *Let $\mathbb{X}$ and $\mathbb{Y}$ be two alphabets, one has*

$$Q_{\lambda}^{(\alpha)}(\mathbb{X} + \mathbb{Y}) = \sum_{\mu} Q_{\mu}^{(\alpha)}(\mathbb{X}) Q_{\lambda/\mu}^{(\alpha)}(\mathbb{Y}),$$

or equivalently

$$P_{\lambda}^{(\alpha)}(\mathbb{X} + \mathbb{Y}) = \sum_{\mu} P_{\mu}^{(\alpha)}(\mathbb{X}) P_{\lambda/\mu}^{(\alpha)}(\mathbb{Y}).$$

Proof See [24] VI.7 for a short proof of this identity. $\square$

An other important normalisation is given by

$$J_{\lambda}^{(\alpha)} = c_{\lambda}(\alpha) P_{\lambda}^{(\alpha)} = c_{\lambda}'(\alpha) Q_{\lambda}^{(\alpha)},$$

where

$$c_{\lambda}(\alpha) = \prod_{(i,j)\in\lambda} (\alpha(\lambda_i - j) + \lambda_j' - i + 1),$$

and

$$c_{\lambda}'(\alpha) = \prod_{(i,j)\in\lambda} (\alpha(\lambda_i - j + 1) + \lambda_j' - i),$$

if λ' denotes the conjugate partition of λ.

If one defines skew J function by

$$J_{\lambda/\mu}^{(\alpha)} := \sum_\nu \frac{\langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)} J_\nu^{(\alpha)} \rangle_\alpha}{\langle J_\nu^{(\alpha)}, J_\nu^{(\alpha)} \rangle_\alpha} J_\nu^{(\alpha)}$$

then $J_{\lambda/\mu}^{(\alpha)}$ is again proportional to $P_{\lambda/\mu}^{(\alpha)}$ and $Q_{\lambda/\mu}^{(\alpha)}$:

$$J_{\lambda/\mu}^{(\alpha)} = c_\lambda(\alpha) c_\mu'(\alpha) P_{\lambda/\mu}^{(\alpha)} = c_\lambda'(\alpha) c_\mu(\alpha) Q_{\lambda/\mu}^{(\alpha)}. \tag{16}$$

4.2 The operator $\int_{\mathbb{Y}}$ and the skew Jack symmetric functions

Let $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$ be three alphabets such that $\sharp\mathbb{X} = n < \infty$ and $\sharp\mathbb{Z} = m < \infty$. Consider the polynomial

$$I_{n,k}(\mathbb{Y}, \mathbb{Z}) = \frac{1}{n!} \int_{\mathbb{Y}} \prod_i x_i^{-m} \prod_{i,j} (x_i + z_j) \Delta^{2k}(\mathbb{X}).$$

Remarking that

$$\begin{aligned} \int_{\mathbb{Y}} x^{p-m} \prod_i (x + z_i) &= \sum_{i=0}^m \Lambda^{p+i-m}(\mathbb{Y}) \Lambda^{m-i}(\mathbb{Z}) \\ &= \Lambda^p(\mathbb{Y} + \mathbb{Z}) \\ &= \int_{\mathbb{Y}+\mathbb{Z}} x^p, \end{aligned}$$

one obtains

$$\begin{aligned} I_{n,k}(\mathbb{Y}, \mathbb{Z}) &= \mathfrak{H}_n^k(\mathbb{Y} + \mathbb{Z}) \\ &= \frac{(-1)^{\frac{kn(n-1)}{2}}}{n!} \binom{nk}{k,\ldots,k} P_{n^k(n-1)}^{(k)}(\mathbb{Y} + \mathbb{Z}). \end{aligned} \tag{17}$$

Hence, the image of $Q_\lambda^{\left(\frac{1}{k}\right)}(\mathbb{X}^\vee) \Delta(\mathbb{X})^{2k}$ by $\int_{\mathbb{Y}}$ is a Jack polynomial.

Corollary 4.2 *One has,*

$$\int_{\mathbb{Y}} Q_\lambda^{\left(\frac{1}{k}\right)}(\mathbb{X}^\vee) \Delta(\mathbb{X})^{2k} = (-1)^{\frac{kn(n-1)}{2}} \binom{nk}{k,\ldots,k} (b_{n^k(n-1)}^{(k)})^{-1} Q_{n^k(n-1)/\lambda'}^{(k)}(\mathbb{Y}),$$

where $b_{n^k(n-1)}^{(k)} = \frac{(2(n-1))!(nk)!((n-1)k)!}{kn!(n-1)!((2n-1)k-1)!}.$

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Proof The equality follows from

$$\prod_i x_i^{-m} \prod_{i,j}(x_i + z_j) = \prod_{i,j}(1 + \frac{z_j}{x_i}) = \sum_\lambda Q_{\lambda'}^{(k)}(\mathbb{Z}) Q_\lambda^{(\frac{1}{k})}(\mathbb{X}^\vee).$$

Indeed, one has

$$I_{n,k}(\mathbb{Y}, \mathbb{Z}) = \frac{1}{n!} \sum_\lambda b_{\lambda'}^{(k)} P_{\lambda'}^{(k)}(\mathbb{Z}) \int_\mathbb{Y} Q_\lambda^{(\frac{1}{k})}(\mathbb{X}^\vee) \Delta(\mathbb{X})^{2k}.$$

And in the other hand, by (17) one obtains

$$I_{n,k}(\mathbb{Y}, \mathbb{Z}) = \frac{(-1)^{\frac{kn(n-1)}{2}}}{n!} \binom{nk}{k,\ldots,k} \sum_\lambda P_\lambda^{(k)}(\mathbb{Z}) P_{n^{k(n-1)}/\lambda}^{(k)}(\mathbb{Y}).$$

Identifying the coefficient of $P_{\lambda'}^{(k)}(\mathbb{Z})$ in the two expressions, one finds,

$$\int_\mathbb{Y} Q_\lambda^{(\frac{1}{k})}(\mathbb{X}^\vee) \Delta(\mathbb{X})^{2k} = (-1)^{\frac{kn(n-1)}{2}} \binom{nk}{k,\ldots,k} (b_\lambda^{(k)})^{-1} P_{n^{k(n-1)}/\lambda'}^{(k)}(\mathbb{Y}),$$

where the value of $b_\lambda^{(k)} := \frac{P_\lambda^{(k)}}{Q_\lambda^{(k)}}$ is given by equality (7). But, from 16, $P_{\lambda/\mu}^{(\alpha)} = \frac{b_\lambda^{(\alpha)}}{b_\mu^{(\alpha)}} Q_{\lambda/\mu}^{(\alpha)}$. Hence,

$$\int_\mathbb{Y} Q_\lambda^{(\frac{1}{k})}(\mathbb{X}^\vee) \Delta(\mathbb{X})^{2k} = (-1)^{\frac{kn(n-1)}{2}} \binom{nk}{k,\ldots,k} (b_{n^{k(n-1)}}^{(k)})^{-1} Q_{n^{k(n-1)}/\lambda'}^{(k)}(\mathbb{Y}).$$

The value of $b_{n^{k(n-1)}}^{(k)}$ is obtained from Equality (7) after simplification. $\square$

5 Even powers of the Vandermonde determinant

5.1 Expansion of the even power of the Vandermonde on the Schur basis

The expansion of even power of the Vandermonde polynomial on the Schur functions is an open problem related the fractional quantum Hall effect as

described by Laughlin's wave function [20]. In particular is of considerable interest to determine for what partitions the coefficients of the Schur functions in the expansion of the square of Vandermonde vanishe [5, 32, 33, 28, 17]. The aim of this subsection is to give an hyperdeterminantal expression for the coefficient of $S_\lambda(\mathbb{X})$ in $\Delta(\mathbb{X})^{2k}$.

Let us denote by $\mathbb{A}_\mathbf{0}$ the alphabet verifying $\Lambda^n(\mathbb{A}_\mathbf{0}) = 0$ for each $n \neq 0$ (and by convention $\Lambda^0(\mathbb{A}_\mathbf{0}) = 1$). The second orthogonality of Jack polynomials can be written as

$$\langle f, g \rangle'_{n,\alpha} = \frac{1}{n!} \int_{\mathbb{A}_\mathbf{0}} f(\mathbb{X}) g(\mathbb{X}^\vee) \prod_{i \neq j} (1 - x_i^{-1} x_j)^{\frac{1}{\alpha}}.$$

In the case when $\alpha = 1$, it coincides with the first scalar product. In particular,

$$\langle S_\lambda(\mathbb{X}), S_\mu(\mathbb{X}) \rangle'_{n,1} = \delta_{\lambda\mu}.$$

Hence, the coefficient of $S_\lambda(\mathbb{X})$ in the expansion of $\Delta(\mathbb{X})^{2k}$ is

$$\langle S_\lambda(\mathbb{X}), \Delta(\mathbb{X})^{2k} \rangle'_{n,1} = \frac{1}{n!} \int_{\mathbb{A}_\mathbf{0}} S_\lambda(\mathbb{X}) \Delta(\mathbb{X}^\vee)^{2k} \prod_{i \neq j} (1 - x_i^{-1} x_j).$$

One has

$$\begin{aligned}
\langle S_\lambda(\mathbb{X}), \Delta(\mathbb{X})^{2k} \rangle'_{n,1} &= (-1)^{\frac{n(n-1)}{2}} \frac{1}{n!} \int_{\mathbb{A}_\mathbf{0}} S_\lambda(\mathbb{X}) \Lambda^n(\mathbb{X})^{(2k+1)(1-n)} \Delta(\mathbb{X})^{2(k+1)} \\
&= (-1)^{\frac{n(n-1)}{2}} \frac{1}{n!} \int_{\mathbb{A}_\mathbf{0}} S_{\lambda + ((2k+1)(1-n))^n}(\mathbb{X}) \Delta(\mathbb{X})^{2(k+1)}.
\end{aligned}$$

By Proposition 2.3, one obtains an hyperdeterminantal expression for the coefficients of the Schur functions in the expansion of the even power of the Vandermonde determinant.

Corollary 5.1 *The coefficient of $S_\lambda(\mathbb{X})$ in the expansion of $\Delta(\mathbb{X})^{2k}$ is the hyperdeterminant*

$$\langle S_\lambda(\mathbb{X}), \Delta(\mathbb{X})^{2k} \rangle'_{n,1} = (-1)^{\frac{n(n-1)}{2}} \mathfrak{H}^{k+1}_{\mathrm{reverse}_n(\lambda) - ((2k+1)(n-1))^n}(\mathbb{A}_\mathbf{0}).$$

It should be interesting to study the link between the notion of admissible partitions introduced by Di Francesco and al [5] and such an hyperdeterminantal expression.

5.2   Jack polynomials over the alphabet $-\mathbb{X}$

In this paragraph, we work with Laurent polynomials in $\mathbb{X} = \{x_1, \cdots, x_n\}$. The space of symmetric Laurent polynomials is spanned by the family indexed by decreasing vectors $(\tilde{S}_\lambda(\mathbb{X}))_{(\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n}$ and defined by

$$\tilde{S}_\lambda(\mathbb{X}) = \frac{\det(x_i^{\lambda_j + n - j})}{\Delta(\mathbb{X})}.$$

Indeed, each symmetric Laurent polynomial f can be written as

$$f(\mathbb{X}) = \Lambda^n(\mathbb{X})^{-m} g(\mathbb{X})$$

where $g(\mathbb{X})$ is a symmetric polynomial in $\mathbb{X}$. As, $g(x)$ is a linear combination of Schur functions, it follows that $f(\mathbb{X})$ is a linear combination of $\tilde{S}_\lambda$'s. Let $\overline{\mathbb{X}} = \{-x_1, \ldots, -x_n\}$ be the alphabet of the inverse of the letters of $\mathbb{X}$. We consider the operation $\int_{-\overline{\mathbb{X}}}$ (i.e. the substitution sending each x^p for $x \in \mathbb{X}$ and $p \in \mathbb{Z}$ to the complete symmetric function $S^n(\mathbb{X})$).

Consider the alternant

$$\begin{aligned}
a_\lambda(\mathbb{X}) &:= \sum_\sigma \epsilon(\sigma) x^{\sigma\lambda} \\
&= \det(x_i^{\lambda_j}) \\
&= \tilde{S}_{\lambda - \delta}(\mathbb{X}) \Delta(\mathbb{X}),
\end{aligned}$$

where $\delta = (n-1, n-2, \ldots, 1, 0)$. From this definition, one obtains that the operator $\frac{1}{n!} \int_{-\overline{\mathbb{X}}}$ sends the product of $2k$ alternants $a_\lambda, a_\mu, \ldots, a_\rho$ is an hyperdeterminant

$$\frac{1}{n!} \int_{-\overline{\mathbb{X}}} a_\lambda(\mathbb{X}) a_\mu(\mathbb{X}) \cdots a_\rho(\mathbb{X}) = \tag{18}$$
$$\mathrm{Det}(S^{\lambda_{i_1} + \mu_{i_2} + \cdots + \rho_{i_{2k i}}}(\mathbb{X}))_{1 \leq i_1, \ldots, i_{2k} \leq n}.$$

Consider the linear operator Ω^+ defined by

$$\Omega^+ \tilde{S}_\lambda(\mathbb{X}) := S_\lambda(\mathbb{X}) := \det(S^{\lambda_i + i - j}(\mathbb{X})). \tag{19}$$

In particular, the operator Ω^+ lets invariant the symmetric polynomials.

Furthermore, it admits an expression involving $\int_{-\overline{\mathbb{X}}}$.

Lemma 5.2 *One has*

$$\Omega^+ = \frac{1}{n!} \int_{-\overline{\mathbb{X}}} a_\delta(\mathbb{X}) a_{-\delta}(\mathbb{X}).$$

Proof It suffices to show that

$$\frac{1}{n!}\int_{-\overline{X}} a_\delta(\mathbb{X})a_{-\delta}(\mathbb{X})\tilde{S}_\lambda(\mathbb{X}) = S_\lambda(\mathbb{X}).$$

But

$$a_\delta(\mathbb{X})a_{-\delta}(\mathbb{X})\tilde{S}_\lambda(\mathbb{X}) = a_{\lambda+\delta}(\mathbb{X})a_{-\delta}(\mathbb{X}),$$

and by (18), one obtains the result. $\square$

Proposition 5.3
Let $\mathbb{X} = \{x_1, \cdots, x_n\}$ be a finite alphabet and $0 \le l \le p \in \mathbb{N}$.

$$R^{(k),n}_{n^p + (k-1)(n-1)l}(-\mathbb{X}) = (-1)^{\frac{(k-1)n(n-1)}{2}+np+l} S_{(p+1)^l p^{n-l}}(\mathbb{X})\Delta(\mathbb{X})^{2(k-1)}. \qquad (20)$$

Proof By Lemma 5.2, one has

$$\begin{aligned}
S_\lambda(\mathbb{X})\Delta(\mathbb{X})^{2(k-1)} &= \Omega^+ S_\lambda(\mathbb{X})\Delta(\mathbb{X})^{(2k-1)} \\
&= \frac{1}{n!}\int_{-\overline{\mathbb{X}}} a_{\lambda+\delta}(\mathbb{X})a_\delta(\mathbb{X})^{2(k-1)}(\mathbb{X})a_{-\delta}(\mathbb{X}).
\end{aligned}$$

By Equality (18), one obtains

$$\begin{aligned}
S_\lambda(\mathbb{X})\Delta(\mathbb{X})^{2(k-1)} &= \mathrm{Det}\left(S^{\lambda_{i_1}+n-i_1+\cdots+n-i_{2k-1}+i_{2k}-n}(\mathbb{X})\right)_{1\le i_1,\ldots,i_{2k}\le n} \\
&= \mathrm{Det}\left(S^{\lambda_{n-i_1}+i_1+\cdots+i_{2k-1}-i_{2k}}(\mathbb{X})\right)_{0\le i_1,\ldots,i_{2k}\le n-1} \\
&= (-1)^{\frac{n(n-1)}{2}}\mathrm{Det}\left(S^{\lambda_{n-i_1}+1-n+i_1+\cdots+i2k}(\mathbb{X})\right)_{0\le i_1,\ldots,i_{2k}\le n-1} \\
&= (-1)^{\frac{n(n-1)}{2}}\mathfrak{H}_{\mathrm{reverse}_n(\lambda)-[(n-1)^n]}(-\overline{\mathbb{X}}) \\
&= (-1)^{\frac{n(n-1)(k-1)}{2}}\mathfrak{T}_{\mathrm{reverse}_n(\lambda)-[((k-1)(n-1))^n]}(-\overline{\mathbb{X}}).
\end{aligned}$$

In particular, from Proposition 3.1

$$\begin{aligned}
R^{(k),n}_{[n^p+(k-1)(n-1)l]}(-\overline{\mathbb{X}}) &= \mathfrak{T}^{(k)}_{[((k-1)(n-1))^n]+[p^{n-l}(p+1)^l]}(-\overline{\mathbb{X}}) \\
&= (-1)^{\frac{n(n-1)(k-1)}{2}}S_{[(p+1)^l p^{n-l}]}(\mathbb{X})\Delta(\mathbb{X})^{2(k-1)}.
\end{aligned}$$

But, the Jack polynomial $R^{(\alpha),n}_\lambda$ being homogeneous, one has

$$R^{(\alpha),n}_\lambda(-\overline{\mathbb{X}}) = (-1)^{|\lambda|}R^{(\alpha),n}_\lambda(-\mathbb{X}).$$

The result follows. $\square$

Remark 5.4   1. Note that a special case of Proposition 5.3 appeared in [22].

2. Proposition 5.3 can be reformulated as

The polynomials $P^{(k)}_{n^p + (k-1)(n-1)l}(-\mathbb{X})$ and $P^{(k)}_{(p^n) + (1^l)}(\mathbb{X})\Delta(\mathbb{X})^{2(k-1)}$ are proportional.

This kind of identities relying Jack polynomials in $\mathbb{X}$ and in $-\mathbb{X}$ can be deduced from more general ones involving Macdonald polynomials when t is specialized to a power of q. This will be investigated in a forthcoming paper.

As a special case of Proposition 5.3, the even powers of the Vandermonde determinants $\Delta(\mathbb{X})^{2k}$ are Jack polynomials on the alphabet $-\mathbb{X}$.

Corollary 5.5 *Setting $l = p = 0$ in Equality (20), one obtains*

$$\Delta(\mathbb{X})^{2k} = \frac{(-1)^{\frac{(kn(n-1)}{2}}}{n!}\begin{pmatrix} (k+1)n \\ k+1, \ldots, k+1 \end{pmatrix} P^{(k+1)}_{n^{(n-1)k}}(-\mathbb{X}).$$

In the same way, using Corollary 4.2, one finds a surprising identity relying Jack polynomials in the alphabets $-\mathbb{X}$ and $\mathbb{X}^\vee$.

Proposition 5.6 *One has*

$$\Omega^+ Q^{(1/k)}_\lambda(\mathbb{X}^\vee)\Delta(\mathbb{X})^{2(k-1)}\Lambda^n(\mathbb{X})^{n-1} =$$
$$\frac{(-1)^{\frac{n(n-1)(k-1)}{2} + |\lambda|}}{n!}\begin{pmatrix} nk \\ k, \ldots, k \end{pmatrix} (b^{(k)}_{n^{k(n-1)}})^{-1} Q^{(k)}_{n^{k(n-1)}/\lambda'}(-\mathbb{X}).$$

Proof The result is a straightforward consequence of Lemma 5.2 and Corollary 4.2. $\square$

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